Group Theory 17

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- **Modular Law** :- A, B, C be subgroups of G with $A \leq B$. If $A \cap C = B \cap C$ and $AC = BC$ then $A = B$

- If A, B and L are subgroups of G with $A \leq L$ then $AB \cap L = A(B \cap L)$ which is a subgroup

Theorem :- **Correspondance Theorem** :-

Let $K \trianglelefteq G$ and $\nu : G \to G/K$ be a natural map. Then $S \to \nu(S) = S/K$ is a bijection from the family of all those subgroups S of G which contain K to the family of all subgroups of G/K.

(i) $T \leq S$ iff $T/K \leq S/K$ and $[S : T] = [S/K : T/K]$

(ii) $T \trianglelefteq S$ iff $T/K \trianglelefteq S/K$ and $S/T \cong \dfrac{S/K}{T/K}$

Q> Let G be a group containing more than 12 elements of order 13. Is G cyclic?

Ans :- Suppose G is cyclic. Let $a \in G$ where $\mathrm{Ord}(a) = 13$

$\left\{ \begin{array}{l} \langle a \rangle \text{ is of finite subgroup of } G. \\ \Rightarrow G \text{ is finite.} \end{array} \right.$

So there will be $\phi(13)$ elements of order 13 in G, i.e., 12.

Hence contradiction.

$\therefore$ G is not cyclic

Hence contradiction.
So G is not cyclic

Q> Let $G = \langle a \rangle$ be a cyclic group and G has a finite subgroup H such that $H \neq \{e\}$. Prove that G is finite.

Ans:- $H = \langle a^m \rangle$ $\quad$ H is finite

$Ord(a^m) = Ord(H) = n$ is finite

$\Rightarrow (a^m)^n = a^{mn} = e \quad \Rightarrow Ord(a) \mid mn \Rightarrow G$ is finite

Q> Let a be an element of a group such that $Ord(a) = n$. Prove that for each $m \geq 1$ we have $\langle a^m \rangle = \langle a^{gcd(m,n)} \rangle$

Ans:- $a^n = e \quad (a^m)^l = e \quad n \mid ml$

Smallest value of l that satisfies it is,

$n / gcd(m,n)$

$$Ord(a^m) = Ord(a^{gcd(m,n)}) = \frac{n}{gcd(m,n)}$$

$\Rightarrow \langle a \rangle$ contains a unique subgroup of order $\frac{n}{gcd(m,n)}$

$\Rightarrow \langle a^m \rangle = \langle a^{gcd(m,n)} \rangle$

Q> If A, B are normal subgroups of group G, then prove that $AB = \{ab : a \in A, b \in B\}$ is also a normal subgroup of G

Ans:- $gAg^{-1} \in A \quad \Rightarrow \quad gABg^{-1} = gAg^{-1}gBg^{-1} \in AB$

$gBg^{-1} \in B$ $\qquad \Rightarrow AB$ is normal

... is cyclic Then prove that

Q> G be a group and $G/Z(G)$ is cyclic Then prove that G is abelian

Ans:— $G/Z(G)$ is cyclic

$G/Z(G) = \langle aZ(G) \rangle \quad a \in G$

Arbitrary $g_1, g_2 \in G$

$g_1 Z(G) = a^n Z(G)$

$g_2 Z(G) = a^m Z(G)$

$g_1 Z(G) g_2 Z(G) = a^n Z(G) a^m Z(G)$
$= a^{m+n} Z(G) = a^m Z(G) a^n Z(G) = g_2 Z(G) g_1 Z(G)$

$g_1 = a^n x \quad x, y \in Z(G)$
$g_2 = a^m y$
$g_1 g_2 = a^n x a^m y = a^m y a^n x = g_2 g_1$
$\Rightarrow G$ is abelian

Q> G is a group such that $Ord(G) = pq$ for some primes p, q. Prove that either $Ord(Z(G)) = 1$ or G is Abelian.

Ans:— Suppose $Ord(Z(G)) < pq$ and $Ord(Z(G)) > 1$.
$Ord(Z(G)) \mid pq$ as $Z(G)$ is a subgroup

$Ord(Z(G)) = p$ or q as p, q are primes
WLOG let $Ord(Z(G)) = p$
Then $Ord(G/Z(G)) = q \Rightarrow G/Z(G)$ is cyclic
$\Rightarrow G$ is abelian.

Then $Ord(G/Z(G)) = q$ $\Rightarrow$ $G/Z(G)$ is cyclic
$\Rightarrow$ G is abelian.

Q> Prove that every subgroup of an Abelian group G is normal

Ans:- $S \leq G$ then S is also abelian
$aS = Sa \quad \forall a \in G$
$\Rightarrow$ S is normal

• > G is a group and $Ord(G) = pq$ for p, q as primes, then,
if $p \nmid q-1$ then $G \cong \mathbb{Z}_{pq}$ and G is cyclic

Q> G is a group of order 105. Prove that $Ord(Z(G))$ is never 7

Ans:- $Ord(G) = 105$
Suppose $Ord(Z(G)) = 7$
$\Rightarrow$ $Ord(G/Z(G)) = 15 = 3 \times 5 = pq$
$p \nmid q-1$ $\Rightarrow$ $G/Z(G)$ is cyclic and $\cong \mathbb{Z}_{3,5}$

$\Rightarrow$ G is abelian

$Ord(Z(G)) = 105$ $\Rightarrow\Leftarrow$